Exercise 22

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x - (-1.5)| < \delta \qquad \text{then} \qquad \left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x + 1.5|$.

$$\left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \varepsilon$$

$$\left| \frac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6 \right| < \varepsilon$$

$$|(3 - 2x) - 6| < \varepsilon$$

$$|-2x - 3| < \varepsilon$$

$$|-2(x + 1.5)| < \varepsilon$$

$$2|x + 1.5| < \varepsilon$$

$$|x + 1.5| < \frac{\varepsilon}{2}$$

Choose $\delta = \varepsilon/2$. Now, assuming that $|x + 1.5| < \delta$,

$$\begin{aligned} \left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| &= \left| \frac{(3 + 2x)(3 - 2x)}{3 + 2x} - 6 \right| \\ &= |(3 - 2x) - 6| \\ &= |-2x - 3| \\ &= |-2(x + 1.5)| \\ &= 2|x + 1.5| \\ &< 2\delta \\ &= 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6.$$